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On the Number of Infinite Geodesics and Ground States in Disordered Systems

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We study first-passage percolation models and their higher dimensional analogs—models of surfaces with random weights. We prove that under very general conditions the number of lines or, in the second case, hypersurfaces which locally minimize the sum of the random weights is with probability one equal to 0 or with probability one equal to $+\infty$. As corollaries we show that in any dimension $d \geq 2$ the number of ground states of an Ising ferromagnet with random coupling constants equals (with probability one) 2 or $+\infty$. Proofs employ simple large-deviation estimates and ergodic arguments.

KEY WORDS: First-passage percolation; disordered systems; geodesics; ground states; minimal hypersurfaces; large deviations.

1. INFINITE GEODESICS IN FIRST PASSAGE PERCOLATION

Let Z^d denote the hypercubic integer lattice in d dimensions. In the first passage percolation model[2,5,10] to each bond of the lattice (i.e., a segment connecting a pair of nearest neighbor sites x and y) there is associated a nonnegative random variable $t_{x,y}$, interpreted as the passage time from x to y. We assume throughout the paper that passage times corresponding to different bonds are independent random variables on some probability space $(\Omega, \mathscr{F}, P)$ with the same continuous distribution. Given a finite nearest neighbor path γ from x to y,

$$\gamma = (x = \gamma_0, \gamma_1, ..., \gamma_k = y) \tag{1}$$

where $\gamma_0, \gamma_1, ..., \gamma_k \in Z^d$, we define the passage time

$$t(\gamma) = \sum_{j=1}^{k} t_{\gamma_{j-1}, \gamma_j} \tag{2}$$

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Proposition 1. Under the above assumptions, with probability one every two nearest neighbor paths connecting the same pair of points have different passage times.

Proof. This follows easily from continuity of the distribution of passage times.

For an arbitrary (not necessarily nearest neighbor) pair (x, y) of sites we define

$$\tau_{x,y} = \inf\{t(\gamma) \mid \gamma \text{ is a path from } x \text{ to } y\} \tag{3}$$

It can be shown[5] that the infimum is actually realized and it follows from Proposition 1 that there is an event $\Omega^* \in \mathscr{F}$ with $P[\Omega^*] = 1$ such that for $\omega \in \Omega^*$ the path which realizes the infimum is unique for any pair (x, y).

Definition 1. For $\omega \in \Omega$ the unique path which realizes the infimum in (3) is denoted by $\gamma_{x,y}$ and called the geodesic connecting x to y.

The geodesics defined above will sometimes be called finite, to distinguish them from infinite geodesics, which we now introduce and which are the main topic of this section.

Definition 2. Let λ be a doubly infinite nearest neighbor path in Z^d, i.e., $\lambda = \{\lambda_n \mid n \in Z\}$ with $|\lambda_{n+1} - \lambda_n| = 1$ for every n. For an $\omega \in \Omega^*$ we call λ an infinite geodesic if for any k, $n \in Z$, $k < n$, the finite path $\{\lambda_m \mid k \leqslant m \leqslant n\}$ is the finite geodesic connecting λ_k to λ_n.

Infinite geodesics are thus doubly infinite paths which locally minimize the passage time. In ref. 8 they are called bigeodesics, to distinguish them from lines which are infinite only in one direction, i.e., $\lambda = \{\lambda_n \mid n \in Z_+\}$, and which are called unigeodesics. We do not consider unigeodesics in this paper.

We denote the number of infinite geodesics by $N(\omega) \in \{0, 1,...\} \cup \{+\infty\}$. The following simple fact is well known.

Proposition 2. $N(\omega)$ is almost surely equal to a constant (finite or infinite).

Proof. $N(\omega)$ does not change under translations of passage time realizations by lattice vectors and is therefore measurable with respect to the tail σ-algebra generated by the variables $t_{x,y}$. The zero–one law implies that it is almost surely constant.

In view of Proposition 2 we will denote the almost surely constant value of $N(\omega)$ by N. The value of N is not known for any continuous distribution of passage times in any dimension. A well-known conjecture says that in $d=2$, $N=0$. For results supporting this conjecture see refs. 8 and 12. The main result of this section, which we present next, is dimension-independent:

Theorem 1. Consider a first passage percolation model in any dimension $d \geqslant 2$. Suppose that the mean passage time for a single bond is finite:

$$E[t_b] < +\infty \tag{4}$$

Then the number of infinite geodesics N equals 0 or $+\infty$.

Proof. Suppose that $0 < N < +\infty$. Let p denote the probability that a given bond belongs to one of the geodesics. It follows from symmetries of the model that this probability does not depend on the choice of the bond. It is also clear that $p > 0$, for otherwise with probability one no bond would belong to any geodesic (since there are countably many bonds) and consequently N would equal zero. Consider a cube Λ_L of linear size L centered at the origin of the lattice. Let $K_L(\omega)$ denote the number of bonds in Λ_L which belong to some geodesic. The multiparameter ergodic theorem[7] implies that

$$\lim_{L \to \infty} \frac{K_L}{|\Lambda_L|} = p \tag{5}$$

for almost every configuration of the passage times. Here and in the sequel $|S|$ denotes the number of elements of the set S. It follows that for any $\varepsilon > 0$ there is an L_0 such that for $L \geqslant L_0$ we have

$$P\left[K_L \geqslant \frac{p}{2} |\Lambda_L| \right] \geqslant 1 - \varepsilon \tag{6}$$

Now, each one of the K_L bonds belongs to (at least) one of the N geodesics, so

$$K_L \geqslant \frac{p}{2} |\Lambda_L| \tag{7}$$

implies that one of the geodesics contains at least $(p/2N)\,|\Lambda_L|$ bonds which lie inside Λ_L; this has to happen with probability at least $1-\varepsilon$. We will show that this leads to a contradiction. The idea of the proof is that if a path takes so many steps to connect two points on the boundary of Λ_L (which is what each geodesic intersecting Λ_L does), then with a probability close to one its passage time exceeds that of a path that connects these two points going along $\partial \Lambda_L$ (the boundary of Λ_L). It follows that the path cannot be a geodesic, contradicting the assumption. To make this argument precise, let us denote the number of bonds on the boundary of Λ_L by s_L; thus s_L is asymptotically equivalent to L^{d-1}. Throughout the rest of the proof let θ be a fixed number with $\theta > E[t_b]$. By the weak law of large numbers[3] we have

$$P\left[\sum_{b \in \partial \Lambda_L} t_b > \theta s_L\right] \to 0 \tag{8}$$

as $L \to \infty$. Next, note that, since for any integer C the number of nearest neighbor walks of length Cs_L starting from 0 is bounded by $(2d)^{Cs_L}$, we have

$$P[\exists \gamma = (0 = \gamma_0, \gamma_1, ..., \gamma_{Cs_L}) \colon t(\gamma) \leqslant \theta s_L] \leqslant (2d)^{Cs_L} P\left[\sum_1^{Cs_L} t_j \leqslant \theta s_L\right] \tag{9}$$

where t_j are independent variables with the distribution of the passage time variables $t_{x,y}$. An application of the Chebyshev inequality[3] yields, for any $r > 0$,

$$P\left[\sum_{j=1}^{Cs_L} t_j \leqslant \theta s_L\right] \leqslant e^{r\theta s_L} E\left[\exp\left(-r \sum_{j=1}^{Cs_L} t_j\right)\right] = [e^{r\theta/C} M(-r)]^{Cs_L} \tag{10}$$

where, by definition, M is the moment-generating function of t_j, i.e.,

$$M(\lambda) \stackrel{\text{def}}{=} E[e^{\lambda t}] \tag{11}$$

Since t_j are positive with probability one, we can choose r such that

$$M(-r) = E[e^{-rt_j}] < \frac{1}{2d} \tag{12}$$

We now choose C so large that

$$2de^{r\theta/C} M(-r) < 1 \tag{13}$$

It follows from the estimate (10) that the probability on its left-hand side decays exponentially fast when $L \to \infty$. Hence also the probability that there exists a path γ originating at a point of $\partial \Delta_L$ with at least Cs_L steps and with the passage time smaller than θs_L decays (exponentially) as $L \to \infty$. Consequently, when L is large, with a probability close to one there are no points x and y on the boundary of Λ_L such that the geodesic

$\gamma_{x,y}$ has more than Cs_L steps. But, we saw, as an immediate consequence of (7), that if $0 < N < \infty$, then with a probability close to one there is a geodesic with at least $(p/2N)\,|\Lambda_L|$ bonds inside Λ_L. Denoting by x and y the points of the first and last intersections of this geodesic with $\partial\Lambda_L$, we obtain a pair of points on the boundary of Λ_L such that the finite geodesic $\gamma_{x,y}$ has at least $(p/2N)\,|\Lambda_L|$ bonds. This leads to the desired contradiction, since for large L

$$\frac{p}{2N}\,|\Lambda_L| > Cs_L \tag{14}$$

End of the proof.

Remark 1. After completing the proof the author was made aware that related arguments had been used to study the number of infinite clusters in percolation models.[10,11] Similar methods are also used in ref. 8.

2. GROUND STATES OF DISORDERED ISING MODEL

In addition to its own interest, the question about the existence and number of infinite geodesics is important for understanding the ground states of disordered ferromagnets, as we now explain.

The Ising model of a ferromagnet is a system of spins $\sigma_x \in \{-1, +1\}$, where $x \in Z^d$, with energy of a configuration $\boldsymbol{\sigma} = \{\sigma_x \mid x \in Z^d\}$ given formally by

$$H(\boldsymbol{\sigma}) = -\sum_{|x-y|=1} J_{x,y}\sigma_x\sigma_y \tag{15}$$

While this sum itself is not well defined, it will be used below to define meaningful expressions. In the standard ferromagnetic Ising model the coupling constants $J_{x,y}$ are all equal to a positive constant J. In models of disordered ferromagnets (i.e., systems whose physics is significantly affected by presence of substitutions, defects, or impurities) the coupling constants are often taken to be random variables. See ref. 4 for a mathematical introduction to the theory of disordered systems. We will consider the case when $J_{x,y}$ are independent nonnegative random variables with a common distribution. This model, called the random exchange Ising model (REIM), has been used to describe ferromagnetic crystals in which some ions have been replaced by ions of a different element, with very similar chemical, but very different magnetic properties. In two dimensions the first passage percolation problem discussed above is equivalent to the problem of the number of ground states of the REIM. From now on we assume that the distribution of the coupling constants $J_{x,y}$ is continuous.

For any pair of spin configurations $\boldsymbol{\sigma}$ and $\boldsymbol{\sigma}'$ such that $\sigma_x = \sigma'_x$ for all except finitely many sites $x \in Z^d$, we define the expression $H(\boldsymbol{\sigma}) - H(\boldsymbol{\sigma}')$ to be the sum of those terms in $H(\boldsymbol{\sigma})$ and $-H(\boldsymbol{\sigma}')$ that do not cancel (there are only finitely many such terms).

Definition 3. For a given realization of exchange coefficients $J_{x,y}$ a spin configuration $\boldsymbol{\sigma}$ is a ground state of the energy function (15) if for any $\boldsymbol{\sigma}'$ equal to $\boldsymbol{\sigma}$ except at finitely many sites we have

$$H(\boldsymbol{\sigma}') - H(\boldsymbol{\sigma}) \geqslant 0 \tag{16}$$

It is clear that the two constant configurations are always the ground states. As for the number of infinite geodesics in the first passage model, a zero–one argument can be used to show that the number of the ground states is almost surely constant (an integer greater than or equal to two, or infinity). In the rest of this section we study the possible number of ground states. We begin by applying theorem of Section 1 to the two-dimensional case.

2.1. Two Dimensions

Theorem 1 has the following interesting corollary:

Theorem 2. In the two-dimensional REIM with continuous distributed independent identically distributed exchange coefficients, the number of the ground states is 2 or $+\infty$.

Proof. Let us introduce the lattice dual to Z^d, i.e., the translation of the original Z^d lattice by the vector $[1/2, 1/2]$. Each bond of the dual lattice is bisecting a unique bond of the original lattice and vice versa, which introduces a one-to-one correspondence between the bonds of the two lattices. We now assign to a bond b of the dual lattice a passage time variable equal to the coupling constant $J_{x,y}$ corresponding to the bond (x, y) of the original lattice which bisects b. This defines a first passage percolation model on the dual lattice. Given a nonconstant ground state (if such ground states exist), consider the bonds of the dual lattice separating $+1$ spins from the -1 ones. It is easy to see that we obtain in this way a union of doubly infinite paths. The definition of a ground state implies immediately that any path separating $+1$ spins from -1 spins in a nonconstant ground state of REIM is an infinite geodesic in the first passage model defined above. Conversely, given an infinite geodesic, we can produce a ground state by defining spins on one side of the geodesic to be

$+1$ and those on the other side to be -1 (there are, of course, two ways to do this; the resulting spin configuration differ by the global sign change). Theorem 2 follows now immediately from Theorem 1.

2.2. Dimensions Higher than Two and Minimal Hypersurfaces

The construction of a dual first-passage percolation model in the previous subsection can be generalized to higher dimensions. Consider a REIM on a lattice Z^d in $d > 2$ dimensions. In dimensions higher than two the objects dual to bonds of Z^d are $(d-1)$-dimensional cells (hypercubes) which bisect the bonds of Z^d. Given a spin configuration on Z^d, we will study the hypersurfaces formed by the cells dual to bonds connecting spins with opposite values. With the cell c dual to a bond (x, y) we associate the random variable $J_c \stackrel{\text{def}}{=} J_{x,y}$—the value of the coupling constant of the corresponding bond. We can think of the resulting model as of a higher dimensional version of first passage percolation, assigning to any finite collection of cells a weight equal to the sum of the weights of its constituent cells. See ref. 6 for results exploring the analogy with first passage percolation. In the present paper we study the analog of infinite geodesics—minimal surfaces—and its application to disordered Ising models. The strategy used in this subsection parallels our approach to the two-dimensional case: we establish a relation between ground states in REIM and minimal hypersurfaces, prove a result about the possible number of minimal hypersurfaces, and, as a corollary, obtain a result about the possible number of ground states.

Definition. The dual of a spin configuration is the union of all cells dual to bonds which connect spins with opposite signs.

It is easy to see that the dual of a spin configuration is a topological manifold without boundary (connected or not). Any orientable, connected topological $(d-1)$-dimensional manifold without boundary which is a union of cells dual to the bonds of Z^d can be obtained as a dual of exactly two spin configurations which differ by the global spin flip.

Definition. Two collections of dual cells are called local modifications of each other if their symmetric difference consists of finitely many cells. Given a configuration of coupling constants $J_{x,y}$ (and hence also the corresponding configuration of weights J_c assigned to the dual cells), for any Σ and its local modification Σ' the expression

$$\sum_{c \in \Sigma'} J_c - \sum_{c \in \Sigma} J_c \tag{17}$$

is well-defined, since the terms of the two sums cancel except for finitely many.

Definition. For a fixed configuration of random variables $J_{x,y}$ an orientable, connected topological $(d-1)$-dimensional manifold Σ which is a union of cells dual to bonds of Z^d is called a minimal hypersurface if it has no local modification Σ' which is also an open, connected topological manifold, such that

$$\sum_{c \in \Sigma'} J_c - \sum_{c \in \Sigma} J_c < 0 \tag{18}$$

As in Proposition 1, it is easy to show that the number of minimal hypersurfaces is almost surely a constant.

Theorem 3. Assume that the random couplings of an REIM are continuously distributed with a finite mean:

$$E[J_{x,y}] < \infty \tag{19}$$

Then in the associated hypersurface model the number of minimal hypersurfaces is 0 or $+\infty$.

The proof of this theorem is very similar to the proof of Theorem 1, so we will only sketch it. The idea is again that if the number N of minimal hypersurfaces is finite and positive, then in a large cube Λ_L centered at the origin, with probability close to one a positive fraction of cells will belong to one of them, which we will denote by Σ. We now construct a local modification of Σ, preserving the infinite components of $\Sigma \backslash \Lambda_L$ and replacing the union of the finite components of $\Sigma \backslash \Lambda_L$ and $\Sigma \cup \Lambda_L$ by a subset of the boundary of Λ_L. This can be done so as to obtain another topological manifold without a boundary, which is a union of cells dual to bonds of the original lattice. A simple way to do this is to consider one of the spin configurations associated to the hypersurface Σ and its local change obtained by putting all spins inside Λ_L to 1. The dual of this new configuration is a local modification of Σ with the desired properties. We now use probabilistic estimates analogous to those of Theorem 1 to show that with probability greater than zero (in fact, close to one),

$$\sum_{c \in \Sigma'} J_c - \sum_{c \in \Sigma} J_c < 0 \tag{20}$$

which is a contradiction.

Theorem 4. Suppose that in a REIM the coupling variables are continuously distributed with a finite mean. Then the number of ground states is 2 or $+\infty$.

Again, the proof is similar to the proof of Theorem 1. It suffices to observe that any minimal hypersurface generates a ground state and that components of the dual configuration corresponding to a ground state are minimal hypersurfaces.

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